

# Pricing of CDOs Based on the Multivariate Wang Transform

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## Abstract

This paper extends the one-factor Gaussian copula model, the standard market model for valuing CDOs, based on the multivariate Wang transform. Unlike the existing models, our model calibrates the parameter associated with a risk adjustment for default threshold, not correlation parameter, which always exists and is unique for any market price of CDO tranche. A Student  $t$ -copula model is also considered within the same framework to describe a fat-tail distribution observed in the actual market. Through numerical experiments, it is shown that our model provides a better fit to the market data compared with the existing models.

*Key words:* One-factor Gaussian copula model, Merton's structural model, multivariate Wang transform, Student  $t$  copula.

*JEL classification:* D81, D92, G32

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## 1. Introduction

The one-factor Gaussian copula (OFGC for short) model, first developed by Li (2000), has become the standard market model for valuing collateralized debt obligations (CDOs) and other basket-type credit derivatives. The Gaussian copula approach is very convenient to model default time correlation given the marginal default probabilities, and allows the semi-analytical form for the pricing and hedging of such products. See, e.g., Hull and White (2004) and Laurent and Gregory (2005) for details of the OFGC model.

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Despite the popularity of the model in practice, however, the copula approach is often criticized due to a couple of reasons.<sup>1</sup> Among them, it is claimed that the copula approach is difficult to interpret and the dependence structure is exogenously given without a theoretical justification. Also, it is well known that the model cannot explain the market prices of CDO tranches, i.e., it exhibits the so-called correlation smile. If the Gaussian copula model fitted market prices well, the implied compound correlation (base correlation, as well) would be approximately constant across tranches.<sup>2</sup> As a result, it cannot price non-standard credit derivatives such as bespoke CDOs to be consistent with market quotes for tranches of standard CDOs.

The aim of this paper is twofold. First, we extend the standard OFGC model in such a way that it has a *sound* interpretation. Namely, we construct an OFGC model based on the multivariate Wang transform developed by Kijima (2006). Second, within the same framework, the Gaussian copula model is further extended to a  $t$ -copula model in order to describe a fat-tail distribution observed in the actual market. Through numerical experiments, it is shown that our model provides a better fit to the market data compared with the existing models.

More specifically, as in CreditMetrics (1997), we start with Merton's structural model (1974) in which underlying firm values follow geometric Brownian motions under the actual probability measure  $P$ . The multivariate Merton model determines the joint default probabilities under  $P$ , which can be converted to those under a pricing measure  $Q$  using the multivariate Wang transform, while keeping the correlation structure as given under  $P$ .<sup>3</sup> The OFGC model is then derived by specifying the correlation structure to be given by  $\rho_{ij} = \rho_i\rho_j$ ,  $-1 < \rho_i, \rho_j < 1$ . The *standard* OFGC model is the

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<sup>1</sup>Given these problems, Mortensen (2006) used a multivariate version of the intensity-based approach to develop a semi-analytical valuation method for CDOs. However, this approach involves many parameters to be estimated and/or calibrated and seems to be computationally difficult and instable to apply for actual markets, compared with the copula approach. See also Duffie and Gârleanu (2001) for the intensity-based approach.

<sup>2</sup>In order to overcome the deficiency, a number of researchers look for copulas that fit market prices better than the Gaussian copula. See, e.g., Hull and White (2006) and Burtschell et al. (2007) for such extensions.

<sup>3</sup>This is consistent with Girsanov's theorem for the Gaussian case in continuous time that, when changing the actual measure to a pricing measure, the mean is adjusted to represent the risk preference of investors, while the variance-covariance structure is unchanged.

one with  $\rho_i = \rho$  for all  $i$ .

We extend the OFGC model by introducing a risk adjustment index for each CDO tranche, since the CDO market is segmented into tranches according to investor's preference against risks. The risk adjustment index is calibrated from the market quote for each CDO tranche, not the correlation parameter, unlike the existing models. It is shown that, under some plausible assumption, the calibrated risk adjustment always exists and is unique for any market price of CDO tranche, which is one of advantages in our model. In the standard copula approach, the existence of implied correlation is not guaranteed in general.<sup>4</sup>

Also, we apply the Student  $t$  copula for the risk-adjusted model, because some empirical studies suggest to use  $t$  distributions with  $\nu = 3$  to 7 degrees of freedom for return distributions of financial and insurance products.<sup>5</sup>

This paper is organized as follows. In the next section, we briefly explain the basis of OFGC models and show that the standard OFGC model can be constructed by applying the multivariate Wang transform. Section 3 is devoted to propose an alternative to the standard OFGC model. A risk adjustment index is introduced for each tranche, which can be interpreted as a risk adjustment for default threshold. Section 4 considers a Student  $t$  copula to describe the fat-tail distribution observed in the actual market, and Section 5 concludes this paper. For the reader's convenience, a concise summary of the Wang transform is provided in Appendix A. The existence and uniqueness of the risk adjustment index is proved under some plausible assumption in Appendix B.

Throughout this paper, we denote the actual probability measure by  $P$  and a pricing measure by  $Q$ . The associated expectation operators are written by  $E$  and  $E^Q$ , respectively. A normal distribution with mean  $\mu$  and variance  $\sigma^2$  is designated by  $N(\mu, \sigma^2)$ . The cumulative distribution function (CDF for short) and probability density function (PDF for short) of a standard normal distribution  $N(0, 1)$  are denoted by  $\Phi(x)$  and  $\phi(x)$ , respectively. The CDF of an  $n$ -variate standard normal distribution with correlation matrix  $\Sigma$  is written as  $\Phi_{n,\Sigma}$ .

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<sup>4</sup>In order to overcome the deficiency, some recent papers such as Baxter (2007) introduce a global catastrophe factor (often called the *Armageddon* factor) with low intensity. That is, when the Armageddon occurs, all the names included in the asset pool default simultaneously.

<sup>5</sup>See, e.g., Platen and Stahl (2003) and Wang (2004) for such empirical studies.

## 2. The OFGC Model

Consider a CDO with an asset pool consisting of  $n$  defaultable assets whose default times are denoted by  $\tau_i$ ,  $i = 1, 2, \dots, n$ . Let  $N_i(t) = 1_{\{\tau_i \leq t\}}$  and  $M_i$  be the default indicator at time  $t$  and the loss given default of name  $i$ , respectively. The time- $t$  cumulative loss of the asset pool is then defined by

$$L(t) = \sum_{i=1}^n M_i N_i(t), \quad 0 \leq t \leq T, \quad (2.1)$$

where  $T$  denotes the maturity of the CDO. Throughout the paper, it is assumed that  $M_i$  are positive constants.

For the pricing of CDOs, we need to know the probability distribution of cumulative loss  $L(t)$  under a pricing measure  $Q$ . To this end, it is essential to model the joint distribution of default times  $\tau_i$ . In this section, we review the one-factor Gaussian copula (OFGC) model, the standard market model for valuing CDOs, within the framework of CreditMetrics (1997) and the multivariate Wang transform.

In order to model the joint distribution of  $(\tau_1, \tau_2, \dots, \tau_n)$  under  $Q$ , we start with the structural model of Merton (1974) under  $P$ . That is, consider the firm value  $V_i(t)$  of name  $i$  that follows the geometric Brownian motion

$$\frac{dV_i(t)}{V_i(t)} = \mu_i dt + \sigma_i dz_i(t), \quad 0 \leq t \leq T; \quad V_i(0) = 1, \quad (2.2)$$

where  $dz_i(t)dz_j(t) = \rho_{ij}dt$ . All the parameters  $\mu_i$ ,  $\sigma_i$  and  $\rho_{ij}$  involved in the model (2.2) are some constants. Note that the firm values are normalized at time zero to have value one.

Now fix  $t$ ,  $0 < t \leq T$ , and suppose that, as in Merton (1974), default occurs before time  $t$  if and only if the firm value  $V_i(t)$  is less than some default threshold. In other words, denoting  $X_i(t) = \log V_i(t)$ , we assume that

$$\{\tau_i \leq t\} = \{X_i(t) < x_i\}, \quad X_i(0) = 0, \quad (2.3)$$

for some threshold  $x_i < 0$ . Note that  $(X_1(t), X_2(t), \dots, X_n(t))$  follows an  $n$ -variate normal distribution with means  $\nu_i t$ , where  $\nu_i = \mu_i - \sigma_i^2/2$ , variances  $\sigma_i^2 t$  and correlation matrix  $\Sigma = (\rho_{ij})$  under  $P$ . Hence, by letting

$$\hat{X}_i = \frac{X_i(t) - \nu_i t}{\sigma_i \sqrt{t}}, \quad i = 1, 2, \dots, n,$$

default probabilities (joint as well as marginals) are defined through

$$\{\tau_i \leq t\} = \{\hat{X}_i < \hat{x}_i(t)\}, \quad \hat{x}_i(t) \equiv \frac{x_i - \nu_i t}{\sigma_i \sqrt{t}}. \quad (2.4)$$

Here,  $(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$  follows an  $n$ -variate *standard* normal distribution with correlation matrix  $\Sigma$  under  $P$ . This means that modeling the probability distribution of cumulative loss  $L(t)$  under  $P$  can be reduced to modeling the joint distribution of  $\hat{X}_i$ . This is the basic idea adopted by CreditMetrics (1997) to evaluate a portfolio credit risk.

**Remark 2.1.** It is worth mentioning that it is computationally very time consuming to obtain the probability distribution of cumulative loss  $L(t)$  for the general correlation case. Hence, CreditMetrics (1997) employs a Monte Carlo simulation to generate sample points of  $(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$  under a given correlation structure in order to approximate the distribution for  $L(t)$ .

In order to determine the joint distribution of  $(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$  under a pricing measure  $Q$ , we adopt the idea of Wang transform, a popular pricing method for financial and insurance risks in the actuarial literature. Recall that  $(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$  follows an  $n$ -variate standard normal distribution with correlation matrix  $\Sigma$  under  $P$ . Hence, according to the multivariate Wang transform (A.5), its joint distribution under  $Q$  also follows an  $n$ -variate normal distribution with the means being adjusted by risk premia  $\theta_i$  and the *same* variance-covariance matrix  $\Sigma$ . In other words, denoting by  $(\hat{X}_1^Q, \hat{X}_2^Q, \dots, \hat{X}_n^Q)$  an  $n$ -variate standard normal random vector with correlation matrix  $\Sigma$  under  $Q$ , it follows from (2.4) that default probabilities (joint as well as marginals) under  $Q$  are defined through

$$\{\tau_i \leq t\} = \{\hat{X}_i^Q < \hat{x}_i^Q(t)\}, \quad \hat{x}_i^Q(t) \equiv \hat{x}_i(t) + \theta_i. \quad (2.5)$$

Hence, the risk premia  $\theta_i$  adjust the default boundaries  $\hat{x}_i(t)$  under the pricing measure  $Q$ .

Now, let us denote by  $F_i(t)$  the *marginal* CDF of  $\tau_i$  under  $Q$ . Recall from (2.5) that we have assumed  $F_i(t) = \Phi(\hat{x}_i^Q(t))$  for each  $t$ , so that

$$\hat{x}_i^Q(t) = \Phi^{-1}(F_i(t)), \quad i = 1, 2, \dots, n. \quad (2.6)$$

That is, given the default CDF  $F_i(t)$  under  $Q$ , the *implied* default threshold for  $\hat{X}_i^Q$  is given by  $\Phi^{-1}(F_i(t))$ . In practice, the implied default threshold is

used for the pricing of CDOs.<sup>6</sup> The CDF  $F_i(t)$  can be calibrated from market quotes for, e.g., credit default swaps (CDSs).<sup>7</sup>

In summary, from (2.5) and (2.6), the joint distribution of  $(\tau_1, \tau_2, \dots, \tau_n)$  under  $Q$  evaluated at time  $t$  is given by

$$F_S(t, t, \dots, t) = \Phi_{n:\Sigma}(\Phi^{-1}(F_1(t)), \Phi^{-1}(F_2(t)), \dots, \Phi^{-1}(F_n(t))), \quad (2.7)$$

where  $F_S(t_1, t_2, \dots, t_n) = Q\{\tau_1 \leq t_1, \tau_2 \leq t_2, \dots, \tau_n \leq t_n\}$ , since  $(\hat{X}_1^Q, \hat{X}_2^Q, \dots, \hat{X}_n^Q)$  follows an  $n$ -variate standard normal distribution with correlation matrix  $\Sigma$ .

**Remark 2.2.** The key assumption to derive the joint distribution (2.7) is the Merton's default criterion (2.3) for each fixed  $t$ . This means that, for distinct  $t$  and  $t'$ , we should consider different underlying models in order to obtain the associated joint distributions, because we cannot define simple default stopping times in this setting. This logical inconsistency can be resolved if alternatively, as in Black and Cox (1976), the default time is defined as the first hitting time to a threshold. However, in the *barrier-style* model, analytical tractability is completely lost and time-consuming Monte Carlo simulation must be employed to determine the joint default distribution. Of course, the *European* counterpart (2.3) will not give identical results to the first-hitting-time model. However, as Baxter (2007) noted, we would be able to calibrate the parameters so that our method matches the barrier-style method closely. These seem the main reasons why the Merton's default criterion (2.3) is used for the pricing of basket-type credit derivatives in many papers despite its logical inconsistency.

**Remark 2.3.** Many papers including Burtschell et al. (2007), Hull and White (2004, 2006) and Laurent and Gregory (2005) for the pricing and hedging of CDOs in the literature start from the modeling of the joint distribution of  $\hat{X}_i^Q$  under  $Q$ . The joint distribution of default times is then constructed by using relation (2.5), i.e. a default criterion of Merton's type.

The industry convention to model the joint distribution of  $(\hat{X}_1^Q, \hat{X}_2^Q, \dots, \hat{X}_n^Q)$  is to employ the OFGC model.<sup>8</sup> More specifically, let  $U$  and  $U_i, i = 1, 2, \dots, n$ ,

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<sup>6</sup>If Merton's model (2.2), or equivalently (2.5), is correct, then the relation between  $\hat{x}_i^Q(t)$  and  $F_i(t)$  will hold as in (2.6).

<sup>7</sup>We use CDSs as basic instruments in the credit derivatives market throughout this paper.

<sup>8</sup>An extension of the model to the multi-factor case is straightforward.

be independent and identically distributed (IID for short)  $N(0, 1)$  random variables under  $Q$ , and define

$$\hat{X}_i^Q = \rho_i U + \sqrt{1 - \rho_i^2} U_i, \quad i = 1, 2, \dots, n, \quad (2.8)$$

where  $\rho_i$ ,  $-1 < \rho_i < 1$ , are some constants. It is readily seen that  $\hat{X}_i^Q$  follow  $N(0, 1)$  with correlation  $\rho_{ij} = \rho_i \rho_j$ ,  $i \neq j$ . In the following, we denote the correlation matrix for this special case by  $\Sigma_\rho$ .

Under these conditions, given the common factor  $U$  in (2.8), the conditional default probability before time  $t$  of name  $i$  under  $Q$  is obtained as

$$q_i(t|U) \equiv Q\{\tau_i \leq t|U\} = Q\{\hat{X}_i^Q \leq \Phi^{-1}(F_i(t))|U\} = \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i U}{\sqrt{1 - \rho_i^2}}\right). \quad (2.9)$$

It follows that the joint CDF of  $(\tau_1, \tau_2, \dots, \tau_n)$  under  $Q$  evaluated at time  $t$  is given by

$$F_S(t, \dots, t) = \int_{-\infty}^{\infty} \left[ \prod_{i=1}^n \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i u}{\sqrt{1 - \rho_i^2}}\right) \right] \phi(u) du. \quad (2.10)$$

Hence, assuming  $\rho_i = \rho$  in (2.10), we recover the standard OFGC model used for the pricing and hedging of CDOs by many practitioners.

**Remark 2.4.** For standard copula models such as the OFGC model (2.10), default correlations among the underlying names are the only unobservable element and, as a result, the correlations play a role of risk premia for investors, which should be calibrated from market quotes for CDO tranches. The industry convention for this purpose is to assume that  $\rho_i = \rho$  for all  $i$ , and the parameter  $\rho$  implied by the market prices, called the *implied correlation*, is used as in much the same way as implied volatilities for the Black–Scholes model. Unfortunately, however, the existence of implied correlation is not guaranteed in general, and this usage of correlation yields an apparent limitation on the model flexibility.

In the OFGC model (2.10), the probability distribution of cumulative loss  $L(t)$  under  $Q$  can be obtained relatively easily. Namely, let  $\varphi(s|t)$  be the characteristic function of  $L(t)$  under  $Q$ . Since  $\hat{X}_i^Q$  are conditionally

independent given the common factor  $U$  and  $\tau_i$  are related to  $\hat{X}_i^Q$  through (2.5) for each  $t$ , we obtain

$$\varphi(s|t) \equiv E^Q [e^{isL(t)}] = E^Q \left[ E^Q \left[ e^{isL(t)} \middle| U \right] \right] = E^Q \left[ \prod_{i=1}^n \varphi_i(s|t, U) \right],$$

where the conditional characteristic function  $\varphi_i(s|t, U)$  is given by

$$\varphi_i(s|t, U) \equiv E^Q \left[ e^{isM_i N_i(t)} \middle| U \right] = q_i(t|U) e^{isM_i} + (1 - q_i(t|U)).$$

Here,  $q_i(t|U)$  is the conditional default probability given by (2.9). It follows that

$$\varphi(s|t) = E^Q \left[ \prod_{i=1}^n \{q_i(t|U) e^{isM_i} + (1 - q_i(t|U))\} \right]. \quad (2.11)$$

The probability distribution of  $L(t)$  can then be numerically inverted back from  $\varphi(s|t)$  using, e.g., the fast Fourier transform.<sup>9</sup>

### 3. A Risk-Adjusted Gaussian Copula Model

We have seen that the standard OFGC model, the model (2.10) with  $\rho_i = \rho$  for all  $i$ , is derived from the Merton's structural model and the multivariate Wang transform with a special correlation structure. However, it is well known that the standard OFGC model cannot explain the market prices of CDO tranches. In this section, we extend the OFGC model (2.10) by introducing a risk adjustment index, called a *base lambda*, for each CDO tranche. The risk adjustment indices are calibrated from market quotes for tranches of standard CDOs. In the following, a base tranche with detachment point  $D$  is called tranche  $D$  for short.

#### 3.1. Risk adjustment

First, note that the joint CDF (2.10) is equal to that of (2.7) with correlation matrix  $\Sigma_\rho$ . That is, the joint CDF of the OFGC model under  $Q$  is alternatively written as

$$F_S(t, t, \dots, t) = \Phi_{n, \Sigma_\rho} \left( \Phi^{-1}(F_1(t)), \Phi^{-1}(F_2(t)), \dots, \Phi^{-1}(F_n(t)) \right). \quad (3.1)$$

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<sup>9</sup>Alternatively, we can apply the bucketing method developed by Hull and White (2004).

Hence, the joint CDF (2.10) can be seen as a special form of the multivariate Wang transform (A.4) with correlation matrix  $\Sigma_\rho$  and no risk adjustment, i.e.  $\lambda = 0$  so that  $\theta_i = 0$  for all  $i$ . Of course, this must be so, because the default thresholds  $\hat{x}_i(t)$  have been adjusted so as to match the market quotes for CDSs. That is, when valuing basket-type credit derivatives, no risk adjustment is required in (3.1) if arbitrage opportunities are precluded across all the credit derivatives markets.

In the actual markets, however, there exists a discrepancy between CDS and CDO markets due to some frictions.<sup>10</sup> For example, as Figure 1 reveals, there exists an inconsistency in the two markets. That is, if there were no market frictions, the market quote of  $[0, 100]$  CDO tranche should coincide with the arithmetic average of CDS spreads, because the  $[0, 100]$  CDO tranche includes all the CDSs as the underlying assets. Also, it is often said that the CDO market itself is segmented into tranches. These frictions may be explained as liquidity risk and/or investor's preference against risks. Hence, it is natural to introduce another risk adjustment index when evaluating each tranche of CDOs in the multivariate Wang transform (3.1).<sup>11</sup> Note that the risk adjustments are to adjust default thresholds so as to match the market quotes for CDOs, as we shall see below.

More precisely, when evaluating tranche  $D$ , we assume that the joint CDF for default times  $\tau_i$  under  $Q$  is given by

$$F_D(t, t, \dots, t) \equiv \Phi_{n, \Sigma_\rho} \left( \Phi^{-1}(F_1(t)) + \lambda_D C_1, \Phi^{-1}(F_2(t)) + \lambda_D C_2, \dots, \Phi^{-1}(F_n(t)) + \lambda_D C_n \right), \quad (3.2)$$

where  $C_i = \sigma_Z \sum_{j=1}^n w_i \rho_{ij}$ ; cf. (A.4) and (3.1). In particular, when the underlying risks are standard normal ( $\hat{X}_i^Q$  in our case), we have

$$C_i \equiv \sum_{j=1}^n \rho_{ij} = 1 - \rho_i^2 + \rho_i \sum_{j=1}^n \rho_j, \quad i = 1, 2, \dots, n, \quad (3.3)$$

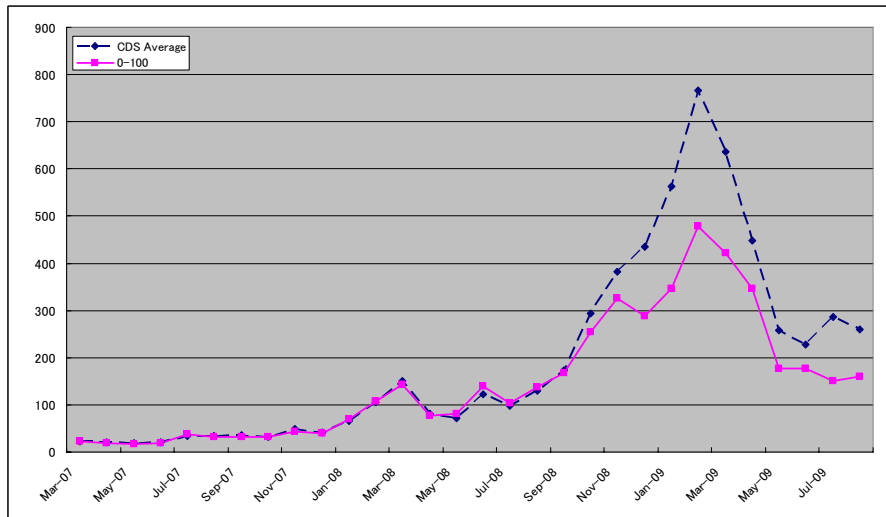
where  $\rho_i$  are defined in (2.8). The parameter  $\lambda_D$  is a risk adjustment index for tranche  $D$  which should be calibrated from market prices.

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<sup>10</sup>Such discrepancy is observed also in e.g. treasury bond, swap and basis swap markets. Some discussions about the discrepancy are found in Kijima et al. (2009) and references therein.

<sup>11</sup>Of course, this yields another logical inconsistency to the no-arbitrage pricing theory. However, the aim of this paper is to develop a practical and statistical method for CDO pricing.

Figure 1: Discrepancy between CDS and CDO markets



While the dashed line depicts an arithmetic average of CDS (mid) spreads, the solid line shows the market quote of [0,100] CDO tranche with the same CDSs as the underlying assets. If there were no market frictions, these values should coincide each other. The CDO [0,100] tranche is traded on the iTraxx Japan index. Those quotes are taken from Bloomberg.

From (3.2) and discussions in the previous section, the associated conditional default probability before time  $t$  of name  $i$  is given by

$$q_{i:D}(t|U) = \Phi \left( \frac{\Phi^{-1}(F_i(t)) + \lambda_D C_i - \rho_i U}{\sqrt{1 - \rho_i^2}} \right); \quad (3.4)$$

cf. (2.9). The associated characteristic function of  $L(t)$ , denoted by  $\varphi_D(s|t)$  is obtained as in (2.11) with the conditional default probability being replaced by  $q_{i:D}(t|U)$ . Note that, when  $\lambda_D = 0$ , the distributions  $F_D(\cdot)$  and  $q_{i:D}(t|U)$  are reduced to  $F_S(\cdot)$  and  $q_i(t|U)$ , respectively.

Our model (3.2) can be interpreted as follows. Let us define

$$x_i^D(t) = \Phi^{-1}(F_i(t)) + \lambda_D C_i, \quad i = 1, 2, \dots, n. \quad (3.5)$$

Whereas  $\Phi^{-1}(F_i(t))$  is the default threshold implied from  $F_i(t)$ , it is readily seen from (3.2) that  $x_i^D(t)$  is the default threshold *implied* from  $F_D(t, t, \dots, t)$ . The risk adjustment index  $\lambda_D$  is a default threshold adjustment for investors of tranche  $D$ . In this paper, while the CDFs  $F_i(t)$  are calibrated from market quotes for CDSs, the risk adjustment index  $\lambda_D$  is calibrated from those for tranche  $D$  of CDOs.

### 3.2. Parameter estimation and calibration

Our model (3.2) involves parameters  $\lambda_D$ ,  $\rho_i$  and the marginal CDFs  $F_i(t)$ ,  $i = 1, 2, \dots, n$ , to be estimated or calibrated from market data.

Calibration is necessary for parameters that reflect risk attitude of investors. In our model, this role is taken by the base lambdas  $\lambda_D$  and the CDFs  $F_i(t)$  of default times  $\tau_i$  under  $Q$ . The calibrated base lambda is called the *implied lambda*. Recall that the CDFs  $F_i(t)$  can be calibrated from market quotes for CDSs.

On the other hand, since we start from Merton's structural model (1974) with the assumption (2.8), i.e.

$$\hat{X}_i^Q = \rho_i U + \sqrt{1 - \rho_i^2} U_i,$$

where  $\hat{X}_i^Q$  denote the standardized log-firm values, it is natural to estimate the correlation parameters  $\rho_i$  using the conventional regression with the standardized stock returns  $X_i$ , if name  $i$  issues a stock, to the standardized return  $U$  of the market index. Note that the correlation structure is the same as the

one under the actual probability measure  $P$ , whence they are to be estimated statistically.

Given the CDFs  $F_i(t)$  of default times and the correlation parameters  $\rho_i$ , the base lambda  $\lambda_D$  can be calibrated from the market quote for tranche  $D$  of standard CDOs. Non-standard credit derivatives such as bespoke CDOs are then priced by using, e.g., appropriate interpolation of the base lambdas. Appendix B shows that the calibrated base lambda  $\lambda_D$  always exists for any CDO market price under some plausible assumption.

### 3.3. A numerical example

In this numerical example, we consider 5 data sets consisting of market quotes for synthetic CDOs and underlying CDSs with the most liquid 5 year maturity. The CDO quotes are available on the five benchmark tranches traded on the Dow Jones iTraxx index, consisting of 125 European investment grade companies, with 0-3%, 3-6%, 6-9%, 9-12% and 12-22% tranches, and iTraxx Japan index, consisting of 80 Japanese investment grade companies, with the same tranches. All quotes are obtained from Bloomberg.

Figure 2 shows the dynamics of tranche spreads of Dow Jones iTraxx index from September 2006 to March 2008. After the collapse of credit derivatives markets on July 2007, the CDO spreads have risen sharply. Some market quotes for the benchmark tranches are listed in Table 1. Accordingly, implied correlations seem to have become extremely high.<sup>12</sup> As usual, strong correlation smiles are observed from the data.

On the contrary to the sharp rise in CDO spreads, Figure 3 reveals that the actual correlations between the underlying assets have dropped after the collapse.<sup>13</sup> Hence, the sharp rise in the CDO spreads cannot be attributed to correlation effects. We should rather think of it as a result of the change in risk attitude of investors. Recall that our model involves risk adjustment parameters  $\lambda_D$  for investors in the market and the correlation structure remains the same as the one under the actual probability measure.

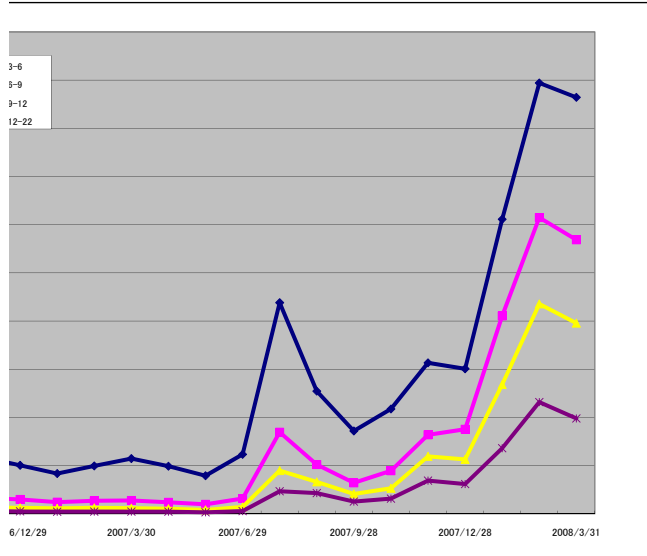
Now, the base lambdas  $\lambda_D$  are calibrated from mid-prices by using a simple bisection method (see Proposition B.2 in Appendix B). Selected curves

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<sup>12</sup>The implied compound correlation for the 9–12% tranche of iTraxx Japan index on March 14, 2008 does not exist for the standard OFGC model.

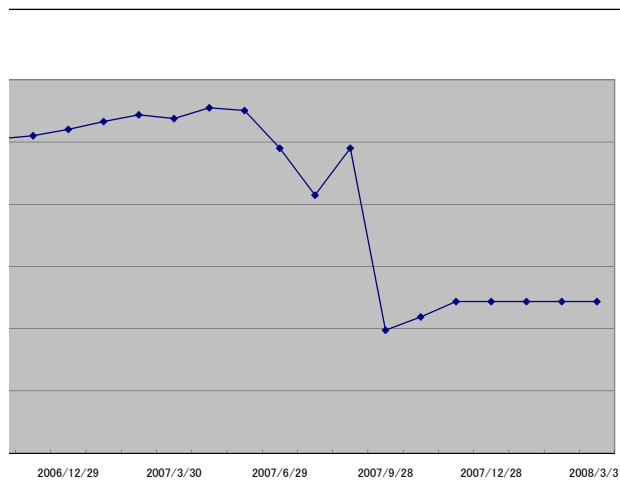
<sup>13</sup>The curve in Figure 3 shows a simple average of correlation parameters  $\rho_i$  calculated by the ordinary regression  $X_i = \rho_i U + \sigma_i U_i$ , where  $X_i$  stands for the standardized stock return of asset  $i$ ,  $U$  the standardized return of market index, and  $U_i$  an error term.

Figure 2: Dynamics of tranche spreads of Dow Jones iTraxx index



The curve shows the dynamics of tranche spreads of Dow Jones iTraxx index from September 2006 to March 2008. After the collapse of the market on July 2007, the CDO spreads have risen sharply.

Figure 3: Averaged correlations of underlying assets



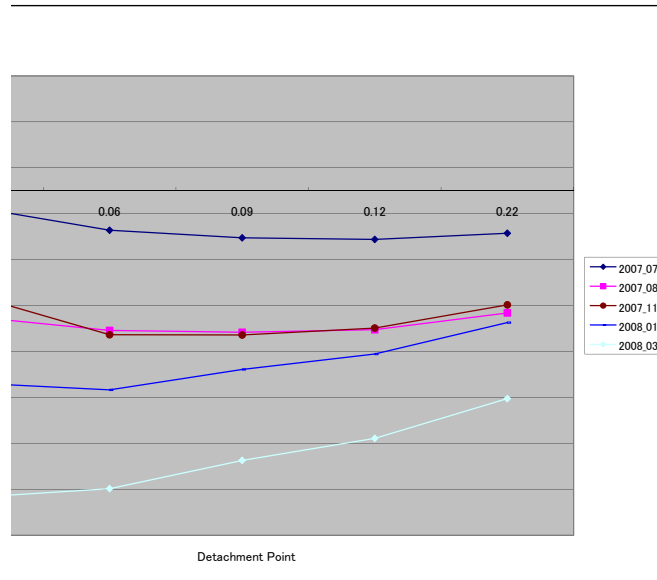
The curve shows an arithmetic average of correlation parameters  $\rho_i$  calculated by the ordinary regression  $X_i = \rho_i U + \sigma_i U_i$ , where  $X_i$  stands for the stock return of asset  $i$ ,  $U$  the return of market index, and  $U_i$  an error term. The regression is performed for monthly data of 5 years.

Table 1: Market and model prices for CDO tranches

Index	Tranches	0-3%	3-6%	6-9%	9-12%	12-22%
DJ iTraxx 8/23/2004	Mid-Price	25.5%	146.0	60.3	36.3	19.3
	Bid/Ask Spread	1.3%	10.0	5.5	5.5	3.5
	Calibrated $\lambda_D$	0.0012	-0.0009	-0.0013	-0.0014	-0.0016
DJ iTraxx 12/05/2005	Mid-Price	26.3%	80.6	23.1	10.3	5.8
	Bid/Ask Spread	0.6%	3.3	2.6	2.0	1.3
	Calibrated $\lambda_D$	0.0031	-0.0002	-0.0012	-0.0016	-0.0020
iTraxx Japan 7/05/2007	Mid-Price	12.2%	40.0	11.0	6.5	3.5
	Bid/Ask Spread	1.5%	10.0	4.0	5.0	3.0
	Calibrated $\lambda_D$	0.0023	-0.0003	-0.0011	-0.0014	-0.0014
iTraxx Japan 7/30/2007	Mid-Price	32.0%	164.0	51.0	31.0	19.8
	Bid/Ask Spread	2.0%	20.0	10.0	10.0	2.0
	Calibrated $\lambda_D$	0.0066	0.0032	0.0019	0.0015	0.0016
iTraxx Japan 12/14/2007	Mid-Price	24.5%	192.5	62.5	30.5	16.5
	Bid/Ask Spread	3.5%	45.0	29.0	15.0	9.0
	Calibrated $\lambda_D$	-0.0042	-0.0043	-0.0046	-0.0047	-0.0042
iTraxx Japan 3/14/2008	Mid-Price	62.8%	1065.0	625.0	315.0	210.0
	Bid/Ask Spread	9.0%	300.0	120.0	100.0	80.0
	Calibrated $\lambda_D$	-0.0103	-0.0086	-0.0077	-0.0078	-0.0062

The market quotes are obtained from Bloomberg. The base lambdas  $\lambda_D$  are calibrated from the mid-spreads based on the joint CDF given by (3.2). The calibration is performed by a simple bisection method. Interest rates are constant at 3%, and the recovery rate is 40%.

Figure 4: Selected base lambda curves



The market quotes of benchmark tranches of Dow Jones iTraxx index are obtained from Bloomberg. The base lambdas  $\lambda_D$  are calibrated from the mid-prices based on the joint CDF given by (3.2). Interest rates are constant at 3%, and the recovery rate is 40%.

of the calibrated base lambdas for Dow Jones iTraxx index are depicted in Figure 4 (see also Table 1). The base lambda curves can take various shapes to represent investor's preference against risks for tranches. In particular, it is interesting to see from Table 1 that, while  $\lambda_D$  are all positive on July 30, 2007 in iTraxx Japan index, they are all negative on December 14, 2007. This is so, because the CDO spreads were relatively wide compared to the CDS spreads on July 30, 2007, while the CDS spreads became wider compared to the CDO spreads on December 14, 2007. Recall that  $\lambda_D C_i$  represents the difference between the implied default thresholds for CDS and CDO values. The same interpretation for the base lambdas applies for the results in Figure 4.

#### 4. A Risk-Adjusted Model with $t$ Copula

The risk-adjusted Gaussian copula model (3.2) has an apparent advantage over the existing models, because it can perfectly fit *any* market quotes for all tranches of standard CDOs by calibrating the base lambdas  $\lambda_D$  (risk adjustment parameters) under some plausible assumption. A non-standard basket credit derivative such as a bespoke CDO can then be priced by using an appropriate base lambda, while keeping the correlation structure as given under the actual probability measure.

However, the model may be difficult to interpret economically, because we consider  $\lambda_D$  as the risk adjustment for tranche  $D$ . Hence, it makes sense economically to assume that the parameter  $\lambda_D$  is increasing in  $D$ , because more risk averse investors will invest higher tranches. For example, a plausible choice for the base lambda  $\lambda_D$  will be

$$\lambda_D = a + b \log D, \quad (4.1)$$

and the parameters  $a$  and  $b$  are to be calibrated. In this section, we consider a Student  $t$  copula to the risk-adjusted model (3.2) to overcome this deficiency.

##### 4.1. Fat-tail distribution

We have seen that the risk-adjusted joint distribution (3.2) for default times  $\tau_i$  can be seen as a multivariate Wang transform. However, it is often said that a drawback of the Wang transform (and its extension to the multivariate setting) is the normal CDF  $\Phi_{n,\Sigma}(\mathbf{x})$  appearing in (3.2), that never matches the fat-tailness observed in the actual market. In fact, some empirical studies suggest to use Student  $t$  distributions, whose CDF is denoted by  $t_\nu(x)$ , with  $\nu = 3$  to  $7$  degrees of freedom for return distributions of financial and insurance assets (see, e.g., Platen and Stahl (2003) and Wang (2004)). In order to overcome this deficiency in (A.1), Wang (2002) proposed the two-parameter transformation

$$G(x) = t_\nu[\Phi^{-1}(F(x)) + \theta], \quad \theta > 0, \quad (4.2)$$

for the pricing of insurance and financial risks and reported that (4.2) is much better to fit to the market data.<sup>14</sup>

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<sup>14</sup>Kijima and Muromachi (2008) showed that the two-parameter transform (4.2) is not consistent with the economic premium principle of Bühlmann (1980).

In our multivariate setting, we also adopt this idea to the joint CDF (3.2) and propose the following multivariate extension of the two-parameter Wang transform:

$$F_D(t, t, \dots, t) = t_{n:\nu, \Sigma_\rho} \left( \Phi^{-1}(F_1(t)) + \lambda_D C_1, \Phi^{-1}(F_2(t)) + \lambda_D C_2, \dots, \Phi^{-1}(F_n(t)) + \lambda_D C_n \right), \quad (4.3)$$

where  $t_{n:\nu, \Sigma_\rho}(\mathbf{x})$  denotes the CDF of an  $n$ -variate standard  $t$  distribution with  $\nu$  degrees of freedom and correlation matrix  $\Sigma_\rho$  for the underlying standard normal random variables.

**Remark 4.1.** Hull and White (2004) suggest to use  $t$  distributions for  $U$  and  $U_i$  in (2.8) under a pricing measure  $Q$ , and reported that the model fits to the market data very well. The good performance of the model is reported in other papers as well. See, e.g., Burtschell et al. (2009) for details. In this model, however, explicit form for the CDF of  $\hat{X}_i^Q$  is not known.

The joint CDF (4.3) can be expressed as follows. Let us define the transformed risks  $X_i^*$  under  $Q$  as

$$X_i^* = \frac{\hat{X}_i^Q}{Y(\nu)}, \quad Y(\nu) = \sqrt{\chi_\nu^2/\nu}, \quad (4.4)$$

where  $\hat{X}_i^Q$  are given by (2.8) and  $\chi_\nu^2$  denotes a random variable that follows a chi-square distribution with  $\nu$  degrees of freedom, independent of other random variables. It is readily checked that the joint CDF (4.3) for default times  $\tau_i$  is given by

$$\begin{aligned} F_D(t, t, \dots, t) &= Q\{X_1^* \leq \Phi^{-1}(F_1(t)) + \lambda_D C_1, X_2^* \leq \Phi^{-1}(F_2(t)) + \lambda_D C_2, \dots, \\ &\quad X_n^* \leq \Phi^{-1}(F_n(t)) + \lambda_D C_n\}. \end{aligned}$$

Now, given the common factor  $U$  in (2.8) and  $Y(\nu)$  in (4.4), the conditional default probability before time  $t$  of name  $i$  is obtained as

$$\begin{aligned} q_{i:D}(t|U, Y(\nu)) &\equiv Q\{X_i^* \leq \Phi^{-1}(F_i(t)) + \lambda_D C_i | U, Y(\nu)\} \\ &= Q\left\{ \rho_i U + \sqrt{1 - \rho_i^2} U_i \leq (\Phi^{-1}(F_i(t)) + \lambda_D C_i) Y(\nu) \mid U, Y(\nu) \right\} \\ &= \Phi\left( \frac{(\Phi^{-1}(F_i(t)) + \lambda_D C_i) Y(\nu) - \rho_i U}{\sqrt{1 - \rho_i^2}} \right). \end{aligned} \quad (4.5)$$

Accordingly, the characteristic function  $\varphi(s|t)$  of  $L(t)$  is obtained as

$$\varphi(s|t) = E^Q \left[ \prod_{i=1}^n \{q_{i:D}(t|U, Y(\nu))e^{isM_i} + (1 - q_{i:D}(t|U, Y(\nu)))\} \right], \quad (4.6)$$

where the expectation is taken over the random variables  $U$  and  $Y(\nu)$ . Hence, the calculation of (4.6) involves a double integration.

#### 4.2. Single integral approximation

In order to avoid the double integration in (4.6),<sup>15</sup> we employ the following approximation instead of (4.4):

$$X_i^* = \frac{\hat{X}_i^Q}{Y_i(\nu)}, \quad Y_i(\nu) = \sqrt{\chi_{i:\nu}^2/\nu}, \quad (4.7)$$

where  $\chi_{i:\nu}^2$  are IID chi-square random variables with  $\nu$  degrees of freedom. Then, we can consider the conditional default probability (4.5) to be given by the common factor  $U$  only.

More specifically, we define random variables  $\xi_i(u)$  by

$$\xi_i(u) = \frac{U_i + \delta_i(u)}{Y_i(\nu)}, \quad \delta_i(u) = \frac{\rho_i u}{\sqrt{1 - \rho_i^2}}, \quad i = 1, 2, \dots, n. \quad (4.8)$$

It is well known that  $\xi_i(u)$  follows a non-central  $t$  distribution with  $\nu$  degrees of freedom and non-centrality parameter  $\delta_i(u)$ . In the following, we denote the CDF of  $\xi_i(u)$  by  $P_{\nu:\delta_i(u)}(x)$ .

Now, given the common factor  $U$  in (2.8), we have from (4.7) and (4.8) that

$$\{X_i^* < \Phi^{-1}(F_i(t)) + \lambda_D C_i\} = \left\{ \xi_i(U) < \frac{\Phi^{-1}(F_i(t)) + \lambda_D C_i}{\sqrt{1 - \rho_i^2}} \right\}.$$

It follows that, given the common factor  $U$ , the conditional default probabil-

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<sup>15</sup>The double integration is computationally time consuming when some parameters are to be calibrated to market data. We aim to construct a simple model for practical use.

ity before time  $t$  of name  $i$  is obtained as

$$\begin{aligned}
q_{i:D}^*(t|U) &\equiv Q\{X_i^* < \Phi^{-1}(F_i(t)) + \lambda_D C_i | U\} \\
&= Q\left\{\xi_i(U) < \frac{\Phi^{-1}(F_i(t)) + \lambda_D C_i}{\sqrt{1 - \rho_i^2}} \middle| U\right\} \\
&= P_{\nu, \delta_i(U)}\left(\frac{\Phi^{-1}(F_i(t)) + \lambda_D C_i}{\sqrt{1 - \rho_i^2}}\right), \quad \delta_i(u) = \frac{\rho_i u}{\sqrt{1 - \rho_i^2}}. \quad (4.9)
\end{aligned}$$

The associated characteristic function of  $L(t)$ , denoted by  $\varphi_D(s|t)$  is obtained as in (2.8) with the conditional default probability being replaced by  $q_{i:D}^*(t|U)$ .<sup>16</sup>

#### 4.3. Empirical application

The model (4.9) involves new parameters  $a$  and  $b$  in (4.1) and  $\nu$ , the degree of freedom in the Student  $t$  distribution. Given the marginal CDFs  $F_i(t)$  and the correlation parameters  $\rho_i$ , these new parameters are calibrated from market quotes for CDO tranches by minimizing the following root mean square price errors (RMSE) relative to bid/ask spreads:

$$\text{RMSE} = \sqrt{\frac{1}{5} \sum_{j=1}^5 \left( \frac{S_{j:\text{mid}} - S_j}{S_{j:\text{ask}} - S_{j:\text{bid}}} \right)^2},$$

where  $S_j$  is the spread of tranche  $j$  calculated by the model,  $S_{j:\text{mid}}$  the market mid-spread,  $S_{j:\text{ask}}$  the market ask spread, and  $S_{j:\text{bid}}$  the market bid spread; see Mortensen (2006) for details.

The calibrated results are shown in Tables 2 and 3, where the calibration results for other models are taken from Mortensen (2006). Our risk-adjusted  $t$ -copula model fits to the market spreads very well, or at least comparable with the existing models. It is interesting to note that the mid-price in equity tranche (higher tranche, respectively) on December 5, 2005 is bigger (smaller) than that on August 23, 2004. In our model, this tranche structure of mid-prices is captured by the degree of freedom  $\nu$  of  $t$  distribution as well as the slope  $b$  of the base lambda  $\lambda_D$  defined in (4.1). More specifically, the

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<sup>16</sup>The CDF  $P_{\nu, \delta_i}(x)$  can be evaluated easily with enough accuracy by using the algorithm developed in Lenth (1989). See Kijima and Muromachi (2008) for details.

slope  $b$  on December 5, 2005 is steeper than that on August 23, 2004, whereas the degree of freedom  $\nu$  on December 5, 2005 is smaller than that on August 23, 2004.

## 5. Conclusion

In this paper, we develop an alternative to the OFGC model, the standard market model for valuing CDOs, by using the multivariate Wang transform to evaluate CDOs and other basket-type credit derivatives. Unlike the existing models, our model calibrates the parameter associated with a risk adjustment for default threshold, not correlation parameter, while keeping the correlation structure as given under the actual probability measure. More precisely, we introduce the base lambda for each tranche to be calibrated from market quotes for CDO tranches, since the CDO market is segmented into tranches according to investor's preference against risks. It is shown that the calibrated base lambda always exists for practical use.

We also consider a Student  $t$  copula model within the same framework, because some empirical studies suggest to use  $t$  distributions with  $\nu = 3$  to 7 degrees of freedom for return distributions of financial and insurance assets. Numerical experiments reveal that our model provide a better fit to the market data than the existing models in the literature.

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## A. The Multivariate Wang Transform

In the actuarial literature, there have been developed many probability transforms for pricing financial and insurance risks. Among them, Wang (2000, 2002) proposed a pricing method based on the following transformation: For a risk with CDF  $F(x)$  under  $P$ , the risk-adjusted CDF  $G(x)$  for the pricing of the risk is given by

$$G(x) = \Phi[\Phi^{-1}(F(x)) + \theta], \quad (\text{A.1})$$

Table 2: Comparison of calibration results for DJ iTraxx index on August 23, 2004

Tranches	0-3%	3-6%	6-9%	9-12%	12-22%	RMSE
Mid-Price	25.5%	146.0	60.3	36.3	19.3	
Bid/Ask Spread	1.3%	10.0	5.5	5.5	3.5	
Jump-diffusion intensities	25.0%	145.1	58.6	38.1	17.7	0.34
Pure diffusion intensities	30.0%	187.1	27.4	3.5	0.1	5.11
Gaussian copula	27.4%	222.3	52.5	13.8	1.6	4.58
EFL Gaussian copula	25.3%	148.9	52.4	43.4	17.9	0.90
Double $t$ copula	24.0%	153.4	56.5	32.4	16.4	0.84
Risk adjusted $t$ copula	25.8%	145.2	49.1	28.4	17.1	1.15

Calibrated parameters:  $\nu = 3$   $\lambda_D = 0.0045 \times \log D - 0.0372$

Table 3: Comparison of calibration results for DJ iTraxx index on December 5, 2005

Tranches	0-3%	3-6%	6-9%	9-12%	12-22%	RMSE
Mid-Price	26.3%	80.6	23.1	10.3	5.8	
Bid/Ask Spread	0.6%	3.3	2.6	2.0	1.3	
Jump-diffusion intensities	28.7%	86.3	18.7	14.4	10.4	2.88
Pure diffusion intensities	32.5%	104.3	8.9	0.8	0.0	6.99
Gaussian copula	34.6%	99.9	2.9	0.1	0.0	8.44
EFL Gaussian copula	27.0%	83.2	9.4	7.4	7.3	2.54
Double $t$ copula	29.8%	101.1	24.4	13.2	6.6	3.99
Risk adjusted $t$ copula	26.5%	77.2	18.5	12.6	8.3	1.37

Calibrated parameters:  $\nu = 1$   $\lambda_D = 0.024 \times \log D - 0.3675$

The market quotes are obtained from Bloomberg. The results in the row ‘Risk adjusted  $t$  copula’ are calibrated from the market quotes based on the model (4.9). The results in the other rows are taken from Mortensen (2006). Interest rates are constant at 3%, and the recovery rate is 40%.

where  $\theta$  is a constant. The transform (A.1) is called the *Wang transform* and the parameter  $\theta$  is considered to be a risk premium. The mean value evaluated under  $G(x)$  will define a risk-adjusted “fair value” of the risk, which can be discounted to time zero using the risk-free interest rate.

The Wang transform not only possesses various desirable properties as a pricing method, but also has a *sound* economic interpretation. For example, as Wang (2003) observed, the transform (A.1) is consistent with Bühlmann’s economic premium principle.

More precisely, Bühlmann (1980) considered risk exchanges among a set of agents. Each agent is characterized by his/her exponential utility function  $u_i(x) = -e^{-\lambda_i x}$ ,  $i = 1, 2, \dots, n$ , and faces a risk of potential loss  $X_i$ . In a pure risk exchange model, Bühlmann (1980) derived the following equilibrium pricing formula for risk  $X$ :<sup>17</sup>

$$\pi(X) = E[\eta X], \quad \eta = \frac{e^{-\lambda Z}}{E[e^{-\lambda Z}]}, \quad (\text{A.2})$$

for which the expectations exist, where  $Z = \sum_{i=1}^n X_i$  is the aggregate risk and  $\lambda$  is given by

$$\lambda^{-1} = \sum_{i=1}^n \lambda_i^{-1}, \quad \lambda_i > 0.$$

The parameter  $\lambda$  is thought of as the *risk aversion index* of the representative agent in the market.

Kijima (2006) extended the Wang transform (A.1) to a multivariate setting based on the Bühlmann’s formula (A.2) as follows. Suppose that the underlying risks are described by an  $n$ -dimensional random vector,  $(X_1, X_2, \dots, X_n)$  say. Suppose further that the underlying risks are formulated by a Gaussian copula. That is, define

$$Z_i \equiv \Phi^{-1}[F_i(X_i)], \quad i = 1, 2, \dots, n,$$

where  $F_i(x)$  is the marginal CDF of  $X_i$ . A Gaussian copula assumes that  $(Z_1, Z_2, \dots, Z_n)$  follows an  $n$ -variate standard normal distribution with correlation matrix  $\Sigma = (\rho_{ij})$ .

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<sup>17</sup>For simplicity, we assume that the risk-free interest rate is zero, or we consider it as a relative price.

Now, as in Wang (2003), suppose that the aggregate risk  $Z$  can be approximated by a normal distribution. Let  $Z_0 = (Z - \mu_Z)/\sigma_Z$  be the standardized normal random variable, where  $\mu_Z = E[Z]$  and  $\sigma_Z^2 = V[Z]$ . Next, suppose that the standard normal variable  $Z_0$  is related to the Gaussian copula as

$$Z_0 \stackrel{d}{=} \xi + \sum_{i=1}^n w_i Z_i, \quad Z_i = \Phi^{-1}[F_i(X_i)], \quad (\text{A.3})$$

for some constants  $w_i$  and random variable  $\xi$ , which is independent of  $Z_i$ , where  $\stackrel{d}{=}$  stands for equality in law. Using the assumption (A.3), Kijima (2006) obtained the following multivariate transformation from the Bühlmann's equilibrium pricing formula (A.2):

$$G(x_1, \dots, x_n) = \Phi_{n:\Sigma} \left( \Phi^{-1}[F_1(x_1)] + \sum_{i=1}^n \lambda_i \rho_{1i}, \dots, \Phi^{-1}[F_n(x_n)] + \sum_{i=1}^n \lambda_i \rho_{ni} \right), \quad (\text{A.4})$$

where  $\lambda_j = \lambda \sigma_Z w_j$ . Note that, when  $n = 1$ , (A.4) coincides with the Wang transform (A.1) with  $\theta = \lambda \sigma_Z w_1$  since  $\rho_{11} = 1$ .

In particular, when the underlying risks  $X_i$  are normally distributed, it can be shown that equation (A.3) always holds with  $w_i = 1/\sigma_Z$  and we have  $G_i(x) = \Phi((x - \mu_i)/\sigma_i)$ , where  $\mu_i = E[X_i]$  and  $\sigma_i^2 = V[X_i]$ . It follows that the multivariate Wang transform (A.4) becomes

$$G(x_1, \dots, x_n) = \Phi_{n:\Sigma} \left( \frac{x_1 - \mu_1}{\sigma_1} + \theta_1, \dots, \frac{x_n - \mu_n}{\sigma_n} + \theta_n \right), \quad (\text{A.5})$$

$$\theta_i \equiv \frac{\lambda}{\sigma_i} \text{Cov}(X_i, Z),$$

since  $\lambda_i = \lambda$  so that the risk premia  $\theta_i$  are given by

$$\theta_i \equiv \sum_{j=1}^n \lambda_j \rho_{ji} = \lambda \sum_{j=1}^n \text{Cov}(Z_i, Z_j).$$

Here,  $\text{Cov}(X, Z)$  is the covariance between  $X$  and  $Z$ ,  $\lambda$  is the risk aversion index, and  $Z$  denotes the aggregated market risk. We note that, in the normal case, the transform (A.5) can be derived directly from the Bühlmann's formula (A.2) without any assumption.<sup>18</sup>

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<sup>18</sup>Kijima (2006) also developed a multivariate version of the Esscher transform and

## B. Existence of Calibrated Base Lambda

For random variables  $X$  and  $Y$ ,  $X$  is said to be greater than  $Y$  in the sense of first-order stochastic dominance (FSD for short), written by  $X \geq_{\text{FSD}} Y$ , if  $F_X(x) \leq F_Y(x)$  for all  $x \in \mathcal{R}$ , where  $F_X(x)$  and  $F_Y(x)$  denote the CDFs of  $X$  and  $Y$ , respectively. It is well known that  $X \geq_{\text{FSD}} Y$  if and only if  $E[f(X)] \geq E[f(Y)]$  for all nondecreasing functions  $f$  for which expectations exist. See, e.g., Kijima and Ohnishi (1996) for details of stochastic dominance relations.

**Proposition B.1.** *Suppose that the joint CDF of default times  $\tau_i$  is given by (3.2). If all  $C_i$  defined in (3.3) are positive, then the cumulative loss  $L(t)$  given by (2.1) is increasing with respect to  $\lambda_D$  in the sense of FSD for each  $t$ .*

*Proof.* First, if  $\tau^a \geq_{\text{FSD}} \tau^b$  then  $N^b(t) \geq_{\text{FSD}} N^a(t)$  for all  $t > 0$ , since  $N(t) = 1_{\{\tau \leq t\}}$  so that we have

$$Q\{N^a(t) = 1\} = Q\{\tau^a \leq t\} \leq Q\{\tau^b \leq t\} = Q\{N^b(t) = 1\}.$$

Second, from (3.4), it is readily seen that, given the common factor  $U$ , each default time  $\tau_i$  is decreasing with respect to  $\lambda_D$  in the sense of FSD. Hence, given  $U$ , each default indicator  $N_i(t)$  is increasing with respect to  $\lambda_D$  in the sense of FSD. But, since FSD satisfies both the convolution and multiplication properties,<sup>19</sup> it is easy to see that, given the common factor  $U$ , the cumulative loss  $L(t) = \sum_{i=1}^n M_i N_i(t)$  is increasing with respect to  $\lambda_D$  in the sense of FSD for each  $t$ . The result follows at once by unconditioning on  $U$ .

For tranche  $[A, D]$ , define

$$h(L) = (L - A)_+ - (L - D)_+, \quad (x)_+ = \max\{x, 0\}. \quad (\text{B.1})$$

The market convention to calculate the annual premium  $S$  of the tranche is the following. Let  $d(t)$  be the discount function, and let  $t_j = j/f$ ,  $j =$

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showed that the multivariate Wang transform (A.5) agrees with the Esscher counterpart for the normal case.

<sup>19</sup>Suppose  $X_1 \geq_{\text{FSD}} Y_1$ ,  $X_2 \geq_{\text{FSD}} Y_2$  and the random variables are independent of each other. The convolution property implies that  $X_1 + X_2 \geq_{\text{FSD}} Y_1 + Y_2$ , whereas the multiplication property means  $wX_1 \geq_{\text{FSD}} wY_1$  for  $w > 0$ . Hence, these properties together ensure that  $w_1X_1 + w_2X_2 \geq_{\text{FSD}} w_1Y_1 + w_2Y_2$  for  $w_1, w_2 > 0$ .

$1, 2, \dots, Tf$ , where  $T$  is the maturity of the CDO and  $f$  is the frequency of coupon payments. Further, define the values of premium leg and protection leg as

$$VL \equiv \frac{S}{f} \sum_{j=1}^{Tf} d(t_j - \hat{f}) \left\{ D - A - \frac{1}{2} E^Q [h(L(t_j)) + h(L(t_{j-1}))] \right\} \quad (\text{B.2})$$

and

$$PL \equiv \sum_{j=1}^{Tf} d(t_j - \hat{f}) E^Q [h(L(t_j)) - h(L(t_{j-1}))] \quad (\text{B.3})$$

with  $L(0) = 0$ , respectively, where  $\hat{f} = (2f)^{-1}$ . The annual premium  $S$  of the tranche is obtained by solving  $VL = PL$ . See, e.g., Mortensen (2006) for details.

Now, the function  $h(L)$  in (B.1) is nondecreasing in  $L$ . It follows from Proposition B.1 that the expectation  $E^Q[h(L(t_j)) + h(L(t_{j-1}))]$  is nondecreasing in  $\lambda_D$  under the condition of the proposition. Also, (B.3) can be rewritten as

$$PL = \sum_{j=1}^{Tf} \{d(t_j - \hat{f}) - d(t_{j+1} - \hat{f})\} E^Q [h(L(t_j))], \quad (\text{B.4})$$

where we set  $d(t_{Tf+1} - \hat{f}) = 0$ . Hence, while  $VL$  in (B.2) is nonincreasing in  $\lambda_D$ ,  $PL$  in (B.3) is nondecreasing in  $\lambda_D$  under the condition of Proposition B.1.<sup>20</sup>

Note that, from (3.4), when  $\lambda_D \rightarrow -\infty$ , we have  $Q\{\tau_i \leq t|U\} \rightarrow 0$  for all  $t > 0$  so that  $\tau_i \rightarrow \infty$  and  $L(t) \rightarrow 0$ . On the other hand, when  $\lambda_D \rightarrow \infty$ , we have  $Q\{\tau_i \leq t|U\} \rightarrow 1$  for all  $t > 0$  so that  $\tau_i \rightarrow 0$  and  $L(t) \rightarrow L \equiv \sum_{i=1}^n M_i$ . It follows that, as  $\lambda_D \rightarrow -\infty$ , we have  $VL \rightarrow \frac{(D-A)S}{f} \sum_{j=1}^{Tf} d(t_j - \hat{f})$  and  $PL \rightarrow 0$ , while  $VL \rightarrow 0$  and  $PL \rightarrow d(t_1 - \hat{f})(D - A)$  as  $\lambda_D \rightarrow \infty$ . The continuity of  $VL$  and  $PL$  with respect to  $\lambda_D$  is obvious.

In particular, for base tranche  $D$  with  $A = 0$ , we have from (B.1) that

$$h(L) = \min\{D, L\}, \quad D - h(L) = (D - L)_+.$$

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<sup>20</sup>We note that, for the ordinary cases, the covariance  $C_i$  defined in (3.3) is usually positive.

When the market price  $S^*$  is observed for tranche  $D$ , the value of the premium leg is calculated as

$$V_D^* \equiv \frac{S^*}{2f} \sum_{j=1}^{Tf} d(t_j - \hat{f}) E^Q [(D - L(t_j))_+ + (D - L(t_{j-1}))_+].$$

Similarly, the value of the protection leg is given by

$$P_D \equiv \sum_{j=1}^{Tf} \{d(t_j - \hat{f}) - d(t_{j+1} - \hat{f})\} E^Q [\min\{D, L(t_j)\}].$$

We want to seek the base lambda  $\lambda_D^*$  that satisfies  $V_D^* = P_D$ . The discussions above guarantee the existence of such  $\lambda_D^*$ .<sup>21</sup> Note that a simple bisection search can be applied to find  $\lambda_D^*$  and it always works.

In summary, we have proved the following result.

**Proposition B.2.** *Suppose that  $D < \sum_{i=1}^n M_i$  and all  $C_i$  are positive. Then, the base lambda  $\lambda_D$  that satisfies  $V_D^* = P_D$  always exists.*

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<sup>21</sup>If there exists an interval  $[a, b]$  such that  $V_D^* = P_D$  for all  $\lambda_D \in [a, b]$  then, by defining, e.g.,  $\lambda_D^* = a$ , we can claim that the calibrated base lambda is unique.

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